Development of the Productive Forces

Gavin Mendel-Gleason
Geoff Hamilton

Dublin City University, School of Computing

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1 Introduction

2 Type Theory

3 Program Transformation
The Problem

CoInductive Nat : Type :=
| Zero  : Nat
| Succ : Nat -> Nat.

CoFixpoint plus (x : Nat) (y : Nat) : Nat :=
match x with
| Zero => y
| Succ x' => Succ (plus x' y)
end.

Require Import List.

CoFixpoint sumlen (xs : list Nat) : Nat :=
match xs with
| nil => Zero
| cons x xs' => Succ (plus x (Succ (sumlen xs'))))
end.

FAIL!

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The Problem

CoInductive Sierp : Set :=
| T : Sierp
| DS : Sierp -> Sierp.

CoInductive CoEq : Sierp -> Sierp -> Prop :=
| coeq_base : CoEq T T
| coeq_next : forall x y, CoEq x y -> CoEq (DS x) (DS y).

CoFixpoint join (x : Sierp) (y : Sierp) : Sierp :=
  match x with
  | T => T
  | DS x' => match y with
    | T => T
    | DS y' => DS (join x' y')
  end
end.

Definition exist (xs : Stream A) (P : A -> Sierp) : Sierp :=
  match xs with
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end.
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FAIL!
data Bool : Set where
  true : Bool
  false : Bool

data Nat : Set where
  zero : Nat
  succ : Nat -> Nat

codata Stream (A : Set) : Set where
  _::_ : A -> Stream A -> Stream A

le : Nat -> Nat -> Bool
le zero _ = true
le _ zero = false
le (succ x) (succ y) = le x y

pred : Nat -> Nat
pred zero = zero
pred (succ x) = x

f : Stream Nat -> Stream Nat
f (x :: y :: xs) = if (le x y)
  then (x :: (f (y :: xs)))
  else (f ((pred x) :: y :: xs))
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What’s going on here?

The problem is that infinite datatypes are not ok unless you can show that they are actually always going to do something and not sit around computing nothing forever. Agda, Coq etc. have amazing type systems that let you prove virtually anything. This means we have to be very careful to avoid proofs which are vacuous.
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- The program

\[
\begin{align*}
  f :: \text{forall } A, A \\
  f = f
\end{align*}
\]

is a problem.
Types

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- Propositions
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Use supercompilation!

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- They can also transform programs which are not type correct into ones that are.
- If we find a systematic way to justify our transformations we can mix type theory and program transformation to correctly type more programs.
- *Bisimulation relations* can give us a uniform justification of proof equivalence which leave the program transformation technique up to the implementer as long as they supply the relation.
- Contexts of free variables

\[ \Gamma \vdash t : A \]
Types

- Contexts of free variables

- A program term

\[ \Gamma \vdash t : A \]
Types

- Contexts of free variables
- A program term
- A type which the program satisfies
Types

- Antecedents

\[ \Gamma_1 \vdash t_1 : A_1 \quad \cdots \quad \Gamma_n \vdash t_n : A_n \]

\[ \Gamma \vdash t : A \]
Types

- Antecedents

$\Gamma_1 \vdash t_1 : A_1 \quad \ldots \quad \Gamma_n \vdash t_n : A_n$

- Consequent

$\Gamma \vdash t : A$
Modus Ponens

- Implication

\[ A \rightarrow B \quad A \quad \text{ImpElim} \]

\[ B \]
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Modus Ponens - AKA: Function Application

- Function
- Proposition
- Consequent

\[ \Gamma \vdash f : A \to B \quad A \to B \]

ImpElim

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Modus Ponens - AKA: Function Application

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- Argument
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\[ \Gamma \vdash f : A \rightarrow B \quad \Gamma \vdash t : A \quad \text{ImpElim} \]

\[ B \]
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- Function
- Argument
- Application

\[ \Gamma \vdash f : A \rightarrow B \quad \Gamma \vdash t : A \]

\[ \Gamma \vdash f \, t : B \]
Proof

- We arrange proof as a tree with every step justified by rules.

\[
\begin{array}{c}
\cdots \\
\frac{\Gamma_1, m \vdash t_{1,m} : A_{1,m}}{\Gamma_1 \vdash t_1 : A_1} & \text{Rule}^i & \cdots & \frac{\Gamma_1, l \vdash t_{1,l} : A_{1,l}}{\Gamma_n \vdash t_n : A_n} & \text{Rule}^j \\
\end{array}
\]
Proof

- We arrange proof as a tree with every step justified by rules.
- Some rules can be terminal

\[ \begin{align*}
\Gamma_1,m &\vdash t_{1,m} : A_{1,m} \\
\hline
\Gamma_1 &\vdash t_1 : A_1 \\
\hline
\Gamma &\vdash t : A
\end{align*} \]

\[ \begin{align*}
\Gamma_1,l &\vdash t_{1,l} : A_{1,l} \\
\hline
\Gamma_n &\vdash t_n : A_n \\
\hline
\Gamma &\vdash t : A
\end{align*} \]
Infinite Proof

- Might we want infinite proofs?

⊢ 1 : N
⊢ 2 : N
...
⊢ map (1+) nats : [N]
⊢ nats : [N] Scons

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Infinite Proof

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\[
\begin{align*}
\vdash 1 : \mathbb{N} & \quad \vdash 2 : \mathbb{N} \\
\vdash map \ (1+) \ nats : [\mathbb{N}] & \quad \text{Scons} \\
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Infinite Proof with a Finite Presentation

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\frac{f : A, \Gamma \vdash Body(f) : A}{\Gamma \vdash f : A} \quad \text{FunRule}
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Recursive Function Rule

Two big problems here

\[
\text{Body}(\omega) = \omega \quad \omega : A \quad \Gamma \vdash \omega : A
\]

FunRule

You are stuck with the recursive form given by your recursive functions.

This will restrict how we can transform proofs (programs!)
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\[
\begin{align*}
\omega : A & \vdash \omega : A \\
\omega : A, \Gamma & \vdash \text{Body}(\omega) : A \\
\Gamma & \vdash \omega : A
\end{align*}
\]  
FunRule
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Cyclic Proofs

Solution?
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- Gives finite presentations of infinite proofs.

*Provided we are careful to put in a condition which ensure soundness.*
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Cyclic Proofs

\[
\frac{\text{case } n' \text{ of} \\
\quad 0 \Rightarrow m \\
\quad | \ S \ n'' \Rightarrow S \ (\text{plus} \ n'' \ m)}{n' : \mathbb{N}, m : \mathbb{N} \vdash \text{plus} \ n' \ m : \mathbb{N}}
\]

\[
\frac{n : \mathbb{N}, m : \mathbb{N} \vdash \text{plus} \ n' \ m : \mathbb{N}}{n : \mathbb{N}, m : \mathbb{N} \vdash S \ (\text{plus} \ n' \ m) : \mathbb{N}}
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\hline
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\text{\hline} \\
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\begin{align*}
&\text{case } n' \text{ of} \\
&\quad 0 \Rightarrow m \\
&\quad \mid S \ n'' \Rightarrow S \ (\text{plus} \ n'' \ m) \\
&\quad \vdash n' : \mathbb{N}, m : \mathbb{N} \vdash \text{plus} \ n' \ m : \mathbb{N} \\
&\quad \vdash n' : \mathbb{N}, m : \mathbb{N} \vdash S \ (\text{plus} \ n' \ m) : \mathbb{N} \\
\end{align*}
\]

\[
\begin{align*}
&\vdash n : \mathbb{N}, m : \mathbb{N} \vdash \text{case } n \text{ of} \\
&\quad 0 \Rightarrow m \\
&\quad \mid S \ n' \Rightarrow S \ (\text{plus} \ n' \ m) \\
&\quad \vdash \theta := \{ (n', n) \}
\end{align*}
\]
(Co)-Termination Requirements

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(Co)-Termination Requirements

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- In order to avoid this with cyclic proof the proof formation rules are not sufficient.
- Something needs to be decreasing for every inductive cycle.
- Some behaviour needs to be ensured for every coinductive cycle.
Inductive Requirements

- Every cycle must have a structurally smaller term.
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- Every cycle returning to the same node must have the same structurally smaller term.

\[
\Gamma' \vdash r : B \quad \Gamma'' \vdash s : C
\]

\[
\vdots
\]

\[
\Gamma \vdash t : A
\]
Inductive Requirements

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The following path: AndIntro\(^2\), OrIntroL\(^1\), chooses the 2nd, and 1st antecedents respectively.

\[
\frac{A}{(A \lor B)} \quad \text{OrIntroL} \\
\frac{C \quad (A \lor B)}{C \land (A \lor B)} \quad \text{AndIntro}
\]
Coinductive Requirements

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$$
\begin{align*}
C & \quad A \\
C \quad (A \lor B) & \quad \text{OrIntroL} \\
C \quad C \land (A \lor B) & \quad \text{AndIntro}
\end{align*}
$$

- A restriction on the form of paths ensures that we can not have *non-productive* computation. That is, all terms will produce some constructor eventually.
Coinductive Requirements

The restriction is made up of two parts, accessible paths, and guarded paths.

Definition (Admissible). A path is called admissible if the first element \( c \) of the path \( p = c, p \) is one of the rule-index-pairs \( \text{OrIntroL}_1, \text{OrIntroR}_1, \text{AndIntro}_1, \text{AndIntro}_2, \text{AllIntro}_1, \alpha\text{Intro}_1, \text{ImpIntro}_1, \text{OrElim}_2, \text{OrElim}_3, \text{AndElim}_2, \text{AllElim}_1, \text{Delta}_1 \) and \( p \) is an admissible path.

Definition (Guardedness). A path is called guarded if it terminates at a Pointer rule, with the sequent having a coinductive type and the path can be partitioned such that \( p = p, [c], p \) where \( c \) is one of the rule-index-pairs \( \text{OrIntroL}_1, \text{OrIntroR}_1, \text{AndIntro}_1, \text{AndIntro}_2, \nu\text{Intro}_1 \) which we will call guards and \( p \) and \( p \) are admissible paths.
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Program Transformation and Cyclic Proof

- Program transformations such as Deforestation and Supercompilation exist naturally in the setting of cyclic proof.
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  - Evaluation steps are always justified.
- We should deem any transformation as appropriate if the resulting term is bisimilar to the original proof.
  - Information propagation.
  - Simplification rules:
    
    $$\begin{align*}
    \text{case} &\quad \begin{cases} 
    x \Rightarrow r \\
    y \Rightarrow s
    \end{cases} & \sim & \begin{cases} 
    x \Rightarrow \text{case } r \text{ of } \\
    y \Rightarrow \text{case } s \text{ of }
    \end{cases} \\
    w \Rightarrow t & \Rightarrow t & w \Rightarrow t & \Rightarrow t
    \end{align*}$$
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- Critically, bisimilar program transformation does not care about non-termination, but it respects it!
- We will neither eliminate nor introduce non-termination.
- This is important because we want to establish that our programs (co)-terminate later, after transformation.
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Program Transformation

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- The stream is filtered by the constraint on the syntax.
- Since the stream is *lazy* we prune branches which will not meet our syntactic restrictions as they are constructed.
The program transformer is *multi-result* - we have a stream of transformed programs.

The stream is filtered by the constraint on the syntax.

Since the stream is *lazy* we prune branches which will not meet our syntactic restrictions as they are constructed.

The streams are implemented with the $\omega$-Monad, a monad which handles the book-keeping of manipulating (potentially) infinite streams.
Success

mutual

\[
\text{sumlen} \_\text{sc} : \text{CoList CoNat} \rightarrow \text{CoNat}
\]
\[
\text{sumlen} \_\text{sc} \ [\ ] = \text{czero}
\]
\[
\text{sumlen} \_\text{sc} \ (x :: xs) = \text{csucc} \ (\text{aux} \ x \ xs)
\]

\[
\text{aux} : \text{CoNat} \rightarrow \text{CoList CoNat} \rightarrow \text{CoNat}
\]
\[
\text{aux} \ \text{czero} \ xs = \text{sumlen} \_\text{sc} \ xs
\]
\[
\text{aux} \ (\text{csucc} \ x) \ xs = \text{csucc} \ (\text{aux} \ x \ xs)
\]
Success

- mutual
  \[
  \text{sumlen}_sc : \text{CoList CoNat} \rightarrow \text{CoNat}
  \]
  \[
  \text{sumlen}_sc \ [\ ] = \text{czero}
  \]
  \[
  \text{sumlen}_sc \ (x :: xs) = \text{csucc} \ (\text{aux} \ x \ xs)
  \]

- Success!
Success?

Definition exist (xs : Stream A) (P : A -> Sierp) : Sierp :=
    match xs with
    | Cons x xs' => join (P x) (exist xs' P)
    end.
Definition exist (xs : Stream A) (P : A -> Sierp) : Sierp :=
    match xs with
    | Cons x xs' => join (P x) (exist xs' P)
    end.

FAIL!

Definition exist (xs : Stream A) (P : A -> Sierp) : Sierp :=
    match xs with
    | Cons x xs' => join (P x) (exist xs' P)
    end.

FAIL!
Success?

- **Definition exist** $(xs : \text{Stream } A) (P : A \rightarrow \text{Sierp}) : \text{Sierp} :=$
  
  \[
  \begin{align*}
  &\text{match } xs \text{ with} \\
  &\quad | \text{Cons } x \ \text{xs'} \Rightarrow \text{join (P x) (exist \text{xs'} P)} \\
  &\text{end.}
  \end{align*}
  \]

- **FAIL!**

- **Why?** Need a result about the associativity of $\text{join}$.
**Success?**

- **Definition** exist (xs : Stream A) (P : A -> Sierp) : Sierp :=
  match xs with
   | Cons x xs' => join (P x) (exist xs' P)
  end.

- **FAIL!**

- **Why?** Need a result about the associativity of `join`.

- **But** supercompilation can do this! Need to use the *right* generalisation and supercompilation on args.
Success?

- Definition exist (xs : Stream A) (P : A → Sierp) : Sierp :=
  match xs with
  | Cons x xs’ => join (P x) (exist xs’ P)
  end.

FAIL!

- Why? Need a result about the associativity of join.
- But supercompilation can do this! Need to use the right generalisation and supercompilation on args.
- Justifiable only if the type theory internally supports bisimulation.
Future Work

- A framework for manipulating a more practical programming language (such as Haskell or Agda).
Future Work

- A framework for manipulating a more practical programming language (such as Haskell or Agda).
- Extension to a type theory with explicit substitution for bisimilar terms.
Future Work

- A framework for manipulating a more practical programming language (such as Haskell or Agda).
- Extension to a type theory with explicit substitution for bisimilar terms.
- A system for the programmers to interactively transform productive terms into syntactically productive terms by using substitution.
The End
To use program transformation with type theory we need justifications that our program transformations are correct.
Bisimulation

- To use program transformation with type theory we need justifications that our program transformations are correct.
- Bisimulations allow us to show equivalences even when we might have infinite behaviours.
To use program transformation with type theory we need justifications that our program transformations are correct.

Bisimulations allow us to show equivalences even when we might have infinite behaviours.

Relies on a coinductive relation between terms.
Bisimulation

- To use program transformation with type theory we need justifications that our program transformations are correct.
- Bisimulations allow us to show equivalences even when we might have infinite behaviours.
- Relies on a coinductive relation between terms.
- We can generate them in the process of program transformation.
Bisimulation

- To use program transformation with type theory we need justifications that our program transformations are correct.
- Bisimulations allow us to show equivalences even when we might have infinite behaviours.
- Relies on a *coinductive* relation between terms.
- We can generate them in the process of program transformation.
- If two terms $s$ and $t$ are bisimilar we write $s \sim t$
Coinduction

- With induction we want to describe a property over each constructor assuming the sub-case.
Coinduction

With induction we want to describe a property over each constructor assuming the sub-case.

\[ P(0) \land P(n) \rightarrow P(n + 1) \]

e.g. the integers, prove \( P(0) \land P(n) \rightarrow P(n + 1) \), to get \( P(n) \)
With induction we want to describe a property over each constructor assuming the sub-case.

E.g. the integers, prove $P \ 0 \land P \ n \rightarrow P(n + 1)$, to get $P \ n$

With coinduction we want to describe a property over each destructor assuming the super-case.
Coinduction

- With induction we want to describe a property over each constructor assuming the sub-case.
e.g. the integers, prove $P \ 0 \land P \ n \Rightarrow P(n + 1)$, to get $P \ n$

- With coinduction we want to describe a property over each destructor assuming the super-case.
e.g. streams, prove $P \ l \Rightarrow P \ (\text{head} \ l) \land P \ (\text{tail} \ l)$ to get $P \ l$
Parks’ Principle

- Useful example of a coinductively defined relation

When are two infinite streams the same?

When every element is the same...

If $\text{head } l = \text{head } l'$ and $\text{tail } l \equiv \text{tail } l'$ assuming $l \equiv l'$

Parks’ Principle

- Useful example of a coinductively defined relation
- When are two infinite streams the same?
Parks’ Principle

- Useful example of a coinductively defined relation
- When are two infinite streams the same? When every element is the same...
Parks’ Principle

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- When are two infinite streams the same? When every element is the same...

\[ l \mathcal{R} l' \]
Parks’ Principle

- Useful example of a coinductively defined relation
- When are two infinite streams the same? When every element is the same...

\[ l \overset{R}{\rightarrow} l' \]

\[ \text{if } \text{head } l = \text{head } l' \]
Parks’ Principle

- Useful example of a coinductively defined relation
- When are two infinite streams the same? When every element is the same...

\[ l \ R \ l' \]
if \( \text{head } l = \text{head } l' \)
and \( \text{tail } l \ R \text{tail } l' \)
Useful example of a coinductively defined relation

When are two infinite streams the same? When every element is the same...

\[ l \, R \, l' \]

\[
\text{if } \ head \ l \ = \ head \ l' \\
\text{and } \ tail \ l \ \, R \ \, tail \ l' \ \text{assuming } \ l \, R \, l'
\]
Bisimulation is formed from two coinductively defined simulation relations:
Simulation

Bisimulation is formed from two coinductively defined simulation relations:

- $s \lesssim t$
Simulation

Bisimulation is formed from two coinductively defined simulation relations:

\[ s \preceq t \land t \preceq s \]
Bisimulation is formed from two coinductively defined simulation relations:

- $s \preceq t \land t \preceq s$
- $s \preceq t$ says that whenever $s \xrightarrow{a} s'$ then $t \xrightarrow{a} t'$ and $s' \preceq t'$
Example Bisimulation
Example Bisimulation

We show one direction of simulation...

\[ u \rightarrow a \rightarrow u' \rightarrow a \rightarrow u'' \]
\[ b \rightarrow a \rightarrow v' \rightarrow a \rightarrow v'' \]
\[ v \rightarrow a \rightarrow v' \rightarrow a \rightarrow v'' \]

\[ v \rightarrow a \rightarrow v' \rightarrow a \rightarrow v'' \]
Example Bisimulation

We show one direction of simulation...

- when $u \xrightarrow{a} u$ we choose $u' \xrightarrow{a} u''$
- and need to show $u R u''$
- assuming $u R u'$
Example Bisimulation

We show one direction of simulation...

- when $u \xrightarrow{a} u$ we choose $u' \xrightarrow{a} u''$ and need to show $u R u''$ assuming $u R u'$
- when $u \xrightarrow{a} u$ we choose $u'' \xrightarrow{a} u'$ and need to show $u R u'$ (done!)
We show one direction of simulation...

- when $u \xrightarrow{a} u$ we choose $u' \xrightarrow{a} u''$ and need to show $u \mathbin{R} u''$ assuming $u \mathbin{R} u'$
- when $u \xrightarrow{b} v$ we choose $u'' \xrightarrow{b} v''$ (done! no further behaviour)
We show one direction of simulation...

- when $u \xrightarrow{a} u$ we choose $u' \xrightarrow{a} u''$ and need to show $u R u''$ assuming $u R u'$
- when $u \xrightarrow{a} v$ we choose $u' \xrightarrow{a} v'$ (done! no further behaviour)
We show one direction of simulation...

- when $u \xrightarrow{a} v$ we choose $u' \xrightarrow{a} v''$
  (done! no further behaviour)
We show one direction of simulation...

- when $u \xrightarrow{b} v$ we choose $u' \xrightarrow{b} v'$
  (done! no further behaviour)
Term Transition Systems

Can we use this for a transition system for terms?
Term Transition Systems

Can we use this for a transition system for terms?

- Yes!
Term Transition Systems

Can we use this for a transition system for terms?

Yes! We can now look at (bi)similarity relations over terms.
Term Transition Systems

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- Yes! We can now look at (bi)similarity relations over terms.
- We can look at Parks’ principle again, with $l$ a stream of $A$s, $l : [A]$
Term Transition Systems

Can we use this for a transition system for terms?

- Yes! We can now look at (bi)similarity relations over terms.
- We can look at Parks’ principle again, with \( I \) a stream of \( A \), \( I : [A] \)

\[
\begin{array}{c}
I \\
/ \\
\text{head} \\
/ \\
\text{tail} \\
\downarrow \\
h
\end{array}
\quad
\begin{array}{c}
I' \\
/ \\
\text{head} \\
/ \\
\text{tail} \\
\downarrow \\
h'
\end{array}
\]

\[
\begin{array}{c}
t \\
\downarrow \\
h \\
\end{array}
\quad
\begin{array}{c}
t' \\
\downarrow \\
h' \\
\end{array}
\]
Can we use this for a transition system for terms?

- Yes! We can now look at (bi)similarity relations over terms.
- We can look at Parks’ principle again, with $l$ a stream of $A$s, $l : [A]$

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Term Transition Systems

Can we use this for a transition system for terms?

- Yes! We can now look at (bi)similarity relations over terms.
- We can look at Parks' principle again, with $l$ a stream of $A$s, $l : [A]$

$$
\begin{array}{c}
R \\
\downarrow\quad l \\
\downarrow\
\downarrow\quad l' \\
\downarrow\
\downarrow\quad head \\
\downarrow\quad \downarrow\quad head \\
tail \\
\downarrow\quad \downarrow \\
h \rightarrow h' \\
\downarrow \\
t' \\
\end{array}
$$
Can we use this for a transition system for terms?

- Yes! We can now look at (bi)similarity relations over terms.
- We can look at Parks' principle again, with \( l \) a stream of \( A \)s, \( l : [A] \)

\[
\begin{array}{c}
R \\
\downarrow \\
= \\
\downarrow
\end{array}
\begin{array}{c}
l \\
\downarrow \\
h \rightarrow h' \\
\downarrow
\end{array}
\begin{array}{c}
\downarrow \\
\downarrow
\end{array}
\begin{array}{c}
\downarrow \\
\downarrow \\
R \\
t \rightarrow t'
\end{array}
\]
Term Transition Systems

How do we know which transition we need?
Term Transition Systems

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- Use a structural operational semantics to define transitions.
Term Transition Systems

How do we know which transition we need?

- Use a structural operational semantics to define transitions.
- Each transition corresponds with an experiment which we obtain from the term language and evaluation relation.
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- Each transition corresponds with an experiment which we obtain from the term language and evaluation relation.
- Experiments consist of terms which will lead to a reduction.
Term Transition Systems

How do we know which transition we need?

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  Application $[\ ] c$
Term Transition Systems

How do we know which transition we need?

- Use a structural operational semantics to define transitions.
- Each transition corresponds with an *experiment* which we obtain from the term language and evaluation relation.
- Experiments consist of terms which will lead to a reduction.
  
  Application \([ \ ] c\)

  Type Application \([ \ ] A\)
Term Transition Systems

How do we know which transition we need?

- Use a structural operational semantics to define transitions.
- Each transition corresponds with an experiment which we obtain from the term language and evaluation relation.
- Experiments consist of terms which will lead to a reduction.
  
  Application \([ ] c\)
  Type Application \([ ] A\)
  Case Elimination \(\text{case}[ ] \text{of } \{ \text{nil} \Rightarrow t \mid (x:xs) \Rightarrow s\}\)
Term Transition Systems

How do we know which transition we need?

- Use a structural operational semantics to define transitions.
- Each transition corresponds with an experiment which we obtain from the term language and evaluation relation.
- Experiments consist of terms which will lead to a reduction.
  
  Application \[ [ ] c \]
  Type Application \[ [ ] A \]
  Case Elimination \[ \text{case[ ] of } \{ \text{nil} \Rightarrow t \mid (x : xs) \Rightarrow s \} \]
  Pair Elimination \[ \text{split[ ] as (x, y) in s} \]
Term Transition Systems

What about function terms?
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- Yes!
Term Transition Systems

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- Yes! Use a test value of the appropriate type to look at behaviour

If $f : A \rightarrow B$?

This approach retains extensionality. That is two functions are the same if they are the same when called with the same arguments.

Term Transition Systems

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Term Transition Systems

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- Yes! Use a test value of the appropriate type to look at behaviour
- If $f : A \rightarrow B$ then $f \xrightarrow{\theta c : A} f \ c$
Term Transition Systems

What about function terms?

- Yes! Use a test value of the appropriate type to look at behaviour
- If $f : A \rightarrow B$ then $f \overset{c:A}{\longrightarrow} f \ c$
- $f \sim g$?
What about function terms?

- Yes! Use a test value of the appropriate type to look at behaviour.

- If \( f : A \rightarrow B \) then \( f @: A \rightarrow f \ c \)

- \( f \sim g \) if whenever \( f @: A \rightarrow f \ c \) and \( g @: A \rightarrow g \ c \) then \( f \ c \sim g \ c \)

- This approach retains *extensionality*. That is two functions are the same if they are the same when called with the same arguments.